Technical Notes

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Free Vibration of a Centrally Clamped Spinning Circular Disk

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Nomenclature

a = hub radius
b = disk radius
h = disk thickness
m = disk material density
p = vibration natural frequency
w = transverse deflection
D = flexural rigidity

A NUMBER of investigators[1–8] have treated the problem of the transverse vibrations of spinning disks under the assumption of the predominance of membrane effects. Lamb and Southwell[1,2] employed an approximate technique to bound the lowest frequency in the case that both bending and membrane effects are important. Mote[9] has employed a Rayleigh-Ritz procedure to study the approximate vibration characteristics of variable thickness disks subjected to general membrane stresses.

The problem treated in this Note, that of transverse vibrations of a spinning, centrally clamped circular disk, is contained in the class of problems treated by Mote. However, the approach here is exact to within the accuracy of the numerical computations, requires no appropriate choices of modal functions, and is free of the usual uncertainties of approximate methods.

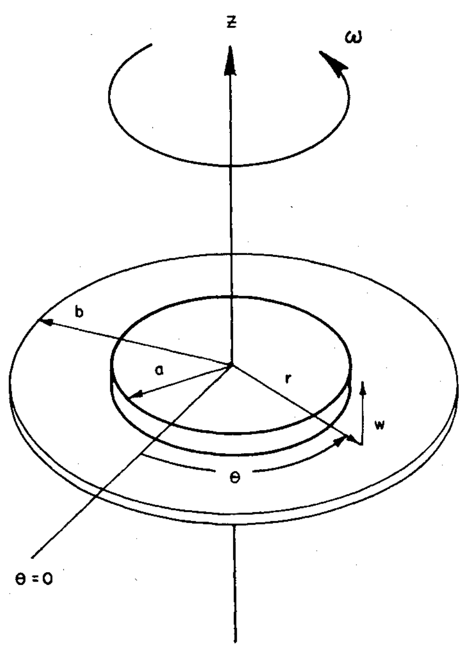

Fig. 1 Configuration of disk with fully clamped hub.

By referring to Fig. 1, consider a uniform, circular disk of radius b spinning about its polar axis with angular velocity ω. The disk is taken to be rigidly clamped on a circle of radius a, and free at the outer edge, $r = b$. At the free edge, $r = b$, the Kirchoff condition and zero bending moment condition must be imposed. At the clamped edge, $r = a$, the deflection and slope must vanish.

For this case, the equilibrium in-plane stresses are given by

$$\sigma_r = (\mu_2/r^2)(b^2 - r^2)(r^2 + \mu_1/b^2\mu_2) \tag{1}$$

$$\sigma_\theta = \frac{\mu_2}{r^2}\left\{\left(b^2 - \frac{\mu_1}{b^2\mu_2}\right)r^2 - \frac{\mu_1}{\mu_2} - \frac{1+3\nu}{3+\nu}r^4\right\} \tag{2}$$

where

$$\mu_1 = \frac{(1-\nu)m\omega^2a^2b^2}{8}\left[\frac{(3+\nu)b^2 - (1+\nu)a^2}{(1+\nu)b^2 + (1-\nu)a^2}\right] \tag{3}$$

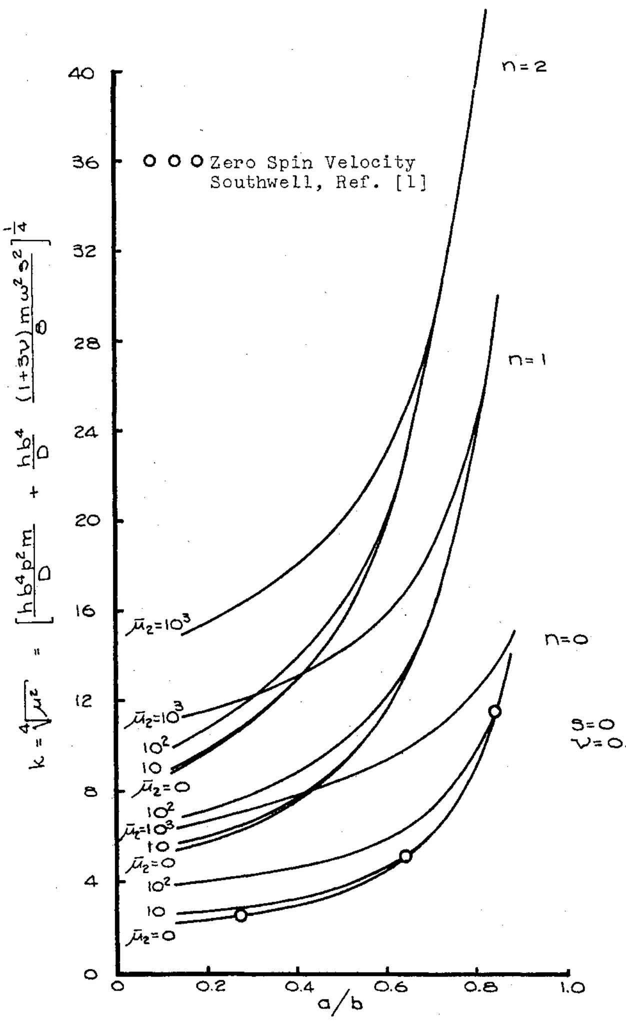

Fig. 2 Fourth root of the frequency parameter μ^2 as a function of the hub to disk radius ratio for zero nodal diameters and from zero to two nodal circles.

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$$\mu_2 = (3 + \nu)m\omega^2/8 \tag{4}$$

The governing differential equation of motion for the transverse vibrations of a spinning circular disk with significant bending rigidity is

$$\frac{D}{h}\nabla^2\nabla^2 w = \frac{1}{r}\frac{\partial}{\partial r}\left(r\sigma_r\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\sigma_\theta\frac{\partial w}{\partial\theta}\right) - m\frac{\partial^2 w}{\partial t^2} \tag{5}$$

After substitution of the in-plane stress distribution and a separation of variables according to

$$w = W(r)\cos s\theta\cos(pt - \epsilon) \tag{6}$$

there is obtained for the radial dependence

$$16x^4\frac{d^4W}{dx^4} + 64x^3\frac{d^3W}{dx^3} + [36 - 4(1 + 2s^2)]x^2\frac{d^2W}{dx^2} + s^2(s^2 - 4)W + 4\bar{\mu}_2\left\{x^2(x - 1)(x + \delta^2)\frac{d^2W}{dx^2} + [2x - (1 - \delta^2)]x^2\frac{dW}{dx} + \frac{1}{4}\left([(1 - \delta^2)x - \delta^2]s^2 - \frac{\mu^2}{\bar{\mu}_2}x^2\right)W\right\} = 0 \tag{7}$$

where

$$x = \left(\frac{r}{b}\right)^2,\ \bar{\mu}_2 = \frac{hb^4}{D}\mu_2 = \frac{hb^4}{D}\frac{(3 + \nu)m\omega^2}{8} \tag{8}$$

$$\mu^2 = \frac{hb^4p^2m}{D} + \frac{hb^4}{D}\frac{(1 + 3\nu)m\omega^2s^2}{8} \tag{9}$$

$$\delta^2 = \frac{\mu_1}{b^4\mu_2} = \frac{(1 - \nu)}{(3 + \nu)}\frac{a^2}{b^2}\left[\frac{(3 + \nu)b^2 - (1 + \nu)a^2}{(1 + \nu)b^2 + (1 - \nu)a^2}\right] \tag{10}$$

The boundary conditions are

$x = (a/b)^2$:

$$W = 0 \tag{11}$$

$$dW/dx = 0 \tag{12}$$

$x = 1$:

$$4x^2\frac{d^2W}{dx^2} + 2x(1 + \nu)\frac{dW}{dx} - s^2\nu W = 0 \tag{13}$$

$$8x^3\frac{d^3W}{dx^3} + \left(\frac{1 + 4\nu}{\nu}\right)4x^2\frac{d^2W}{dx^2} + \left[\left(\frac{1 + \nu}{\nu}\right) - (2 - \nu)s^2\right]2x\frac{dW}{dx} + s^2(2 - \nu)W = 0 \tag{14}$$

To avoid difficulties associated with expansion about the singular point at $x = 0$, solutions are obtained by expansion in power series about an ordinary point of Eq. (7) at the free edge of the disk, $x = 1$. A solution of Eq. (7) can be expanded in the form

$$W = \sum_{k=0}^{\infty} C_k\left[\frac{1 - x}{1 - (a/b)^2}\right]^k \tag{15}$$

The C_k are defined by recurrence relations of the form

$$C_0, C_1, C_2, C_3 = \text{arbitrary}$$

$$\alpha_4C_4 - \beta_4C_3 + \gamma_4C_2 - \delta_4C_1 + \epsilon_4C_0 = 0$$

$$\alpha_5C_5 - \beta_5C_4 + \gamma_5C_3 - \delta_5C_2 + \epsilon_5C_1 - \eta_5C_0 = 0 \tag{16}$$

$$\alpha_kC_k - \beta_kC_{k-1} + \gamma_kC_{k-2} - \delta_kC_{k-3} + \epsilon_kC_{k-4} - \eta_kC_{k-5} + \lambda_kC_{k-6} = 0\ \ k \geq 6$$

where α_k through λ_k are determined by standard techniques.

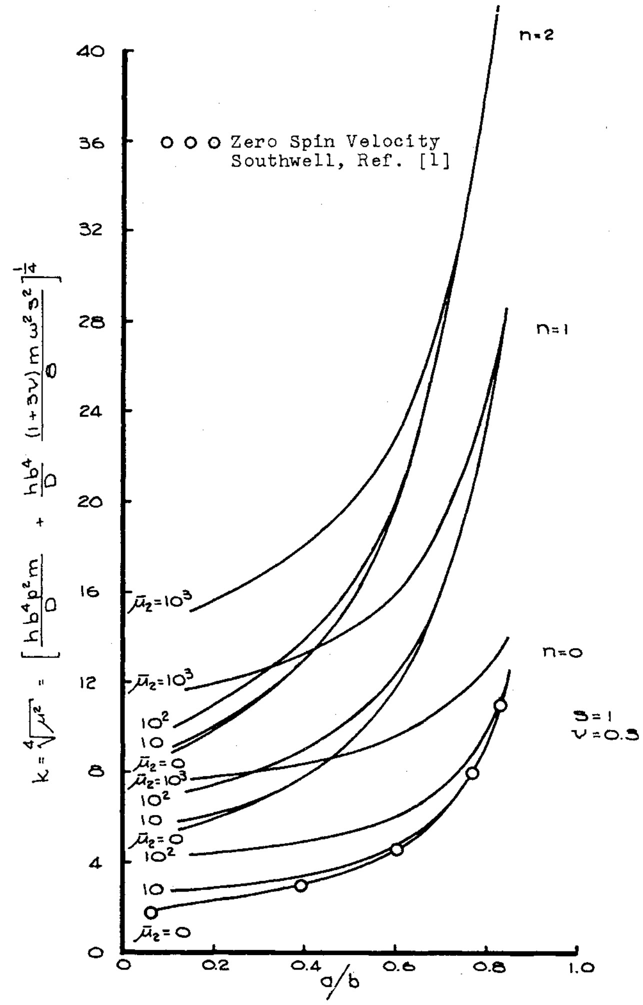

Fig. 3 Fourth root of the frequency parameter μ^2 as a function of the hub to disk radius ratio for one nodal diameter and from zero to two nodal circles.

Since the first four coefficients are arbitrary, four linearly independent solutions can be defined such that the total solution is in the form

$$W = AW_0 + BW_1 + CW_2 + DW_3 \tag{17}$$

where W_i is defined as the solution obtained by setting $C_i = 1$ and the remaining first four $C_j = 0$. The constants A, B, C, and D can be determined by requiring that the four boundary conditions be satisfied. The requirement that the boundary conditions be satisfied results in four homogeneous algebraic equations for the unknown coefficients A, B, C, and D. This system of equations will have a nontrivial solution if the determinant of coefficients is zero. Since these coefficients are a function of the frequency parameter μ^2, and hence a function of the natural frequencies of vibration, the eigenvalue problem requires finding the zeroes of the determinant of coefficients as a function of the parameter μ^2.

Solutions of the eigenvalue equation were found numerically and are plotted in Figs. 2, 3, and 4 as a function of hub to disk ratio a/b, for from zero to two nodal diameters, $s = 0, 1, 2$, and from zero to two nodal circles, $n = 0, 1, 2$. Computations were carried out for a value of Poisson's ratio $\nu = 0.3$. The spin rigidity parameter $\bar{\mu}_2$ was incremented through a range from $\bar{\mu}_2 = 0$ to $\bar{\mu}_2 = 1000$.

The spin rigidity parameter $\bar{\mu}_2$, defined by Eq. (8), scales the relative importance of the membrane effects and the bending effects. For large $\bar{\mu}_2$ the membrane effects predominate and for small $\bar{\mu}_2$ the bending effects predominate. Figures 2, 3, and 4 show results for the case $\bar{\mu}_2 = 0$ from

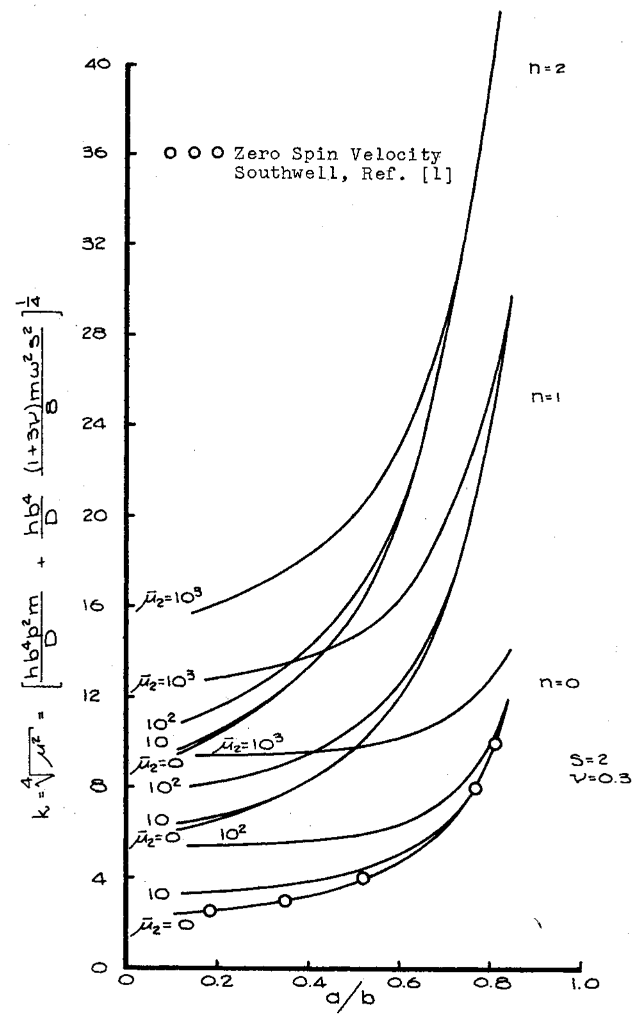

Fig. 4 Fourth root of the frequency parameter μ^2 as a function of the hub to disk radius ratio for two nodal diameters and from zero to two nodal circles.

both the present analysis and from Southwell's[2] work and the agreement is exact to within acceptable computational errors. The cases of large $\bar{\mu}_2$ were compared with Eversman's[8] work and it was found that for $\bar{\mu}_2 \rightarrow \infty$ the frequency curves approached those of the membrane. However, carrying this limiting process to extremely high values of $\bar{\mu}_2$ is difficult because the differential equations and boundary conditions obtained in the limit are of different character than those for finite values of $\bar{\mu}_2$.

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Bounds on the Motion of Objects Ejected from an Orbiting Body

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Introduction

THE purpose of this Note is to present a procedure for determining bounds on the position of objects ejected from an orbiting body or a space vehicle relative to the vehicle trajectory's position. The results are valid for any velocity and direction of ejection. The same analysis can be used to determine an estimate of the predicted vehicle trajectory for a radar observation velocity error in an arbitrary direction. Both problems are described by the same set of linear differential equations with varying coefficients. However, these equations cannot be reduced to a closed form of solution. As a result, the usual analysis has been either to replace the varying coefficients by their average constant values or to make use of computer solutions. In this Note, reasonable bounds on the object's position are obtained by the use of an integral inequality without approximating the solution of the exact or averaged differential equations.

Equations of Motion

The motion of an object in an inverse square gravitational field is given by the differential equations

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 \cos^2\phi - r\left(\frac{d\phi}{dt}\right)^2 = \frac{-g_0R_0^2}{r^2} \qquad (1)$$

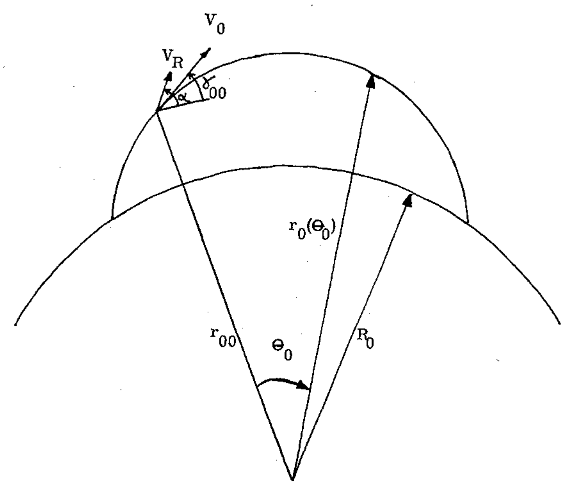

Fig. 1 Trajectory diagram.

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